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Computational Assessment for the Heat Generation Constant $\dot{e}_{gen}$ on three-Dimensional Time-Fractional Nonlinear Heat Equation in a Cylindrical Coordinate

[1*]Falade K. I, [2]Nuru M, [3]Abubakar A,[4]Tanko A.
[1*,2]Department of Mathematics, Faculty of Computing and Mathematical Sciences,
Aliko Dangote University of Science and Technology Wudil, Kano State Nigeria.
[3]Department of Physical Science, Mathematics Unit, Rabiu Musa Kwankwaso College of Advanced and Remedial Studies, Tudun Wada, Kano State Nigeria.
[4]Department of Statistics, Federal College of Forestry, Jos Nigeria

*Corresponding author email: faladekazeem2016@kustwudil.edu.ng

ABSTRACT:
In this article, we present and employ an efficient algorithm for the assessment of the rate of heat generation constant $\dot{e}_{gen[j]}$ on the time-fraction nonlinear three-dimensional heat equation in a cylindrical coordinate. The usage of the Maple software was deployed to formulate a five-step algorithm using the definition of the Riemann-Liouville fractional integral. Five numerical experiments were carried out for different heat generation constant $\dot{e}_{gen[j]}$ and the results are presented in tabular form and 3D plots. From our computational solutions obtained, we observed that an increase in heat generation constant $\dot{e}_{gen[j]}$, yielded more heat generation in respective of the time-fractional order of the heat equation.

Keywords: Four steps algorithm, heat generation constant, time-fractional, nonlinear heat equation, cylindrical coordinate

INTRODUCTION

Heat analysis plays a significant role in the study of thermodynamics and fluid mechanics in the area of chemical and mechanical engineering. It led to heat conduction equations being developed using an energy balance on a differential element inside the medium and remained the same regardless of the thermal conditions on the surfaces of the medium. Generally, heat transfer through a medium is three-dimensional. That is, the temperature varies along all three primary directions within the medium during the heat transfer process and the phenomenon is called heat generation, it occurs throughout the body of a medium which measures the rate of heat generation in a medium and is denoted as $\dot{e}_{gen[j]}$ constant (Ahmed et al., 2020).

In this paper, we aim to implement an efficient algorithm to assess the rate of heat generation constant $\dot{e}_{gen[j]}$ on three-dimensional cylindrical time-fractional nonlinear heat conduction equation in a cylindrical coordinate of the form:

$$\begin{cases} \dfrac{\partial^\alpha \psi}{\partial t^\alpha} = \beta\left(\psi\dfrac{\partial^2\psi}{\partial r^2} + \dfrac{\partial^2\psi}{\partial \varphi^2} + \dfrac{\partial^2\psi}{\partial z^2} + \dfrac{\dot{e}_{gen[j]}}{k}\right), \\ \psi(r,\varphi,z,0) = f(r,\varphi,z), \\ \psi_t(r,\varphi,z,0) = g(r,\varphi,z), \\ 0 < \alpha \le 1, \quad j = 1,2,\dots,5\,. \end{cases} \tag{1}$$

Where $\dot{e}_{gen[j]}$ is the constant rate of heat generation per unit volume, $\beta = \frac{k}{\rho c}$ is the thermal diffusivity of the material, k is thermal conductivity, ρ is the

density of the material, c is the specific heat, and r, φ, z are the cylindrical coordinate points.
We considered the energy generation from an energy balance on a volume element in three-dimensional cylindrical coordinates of the form:

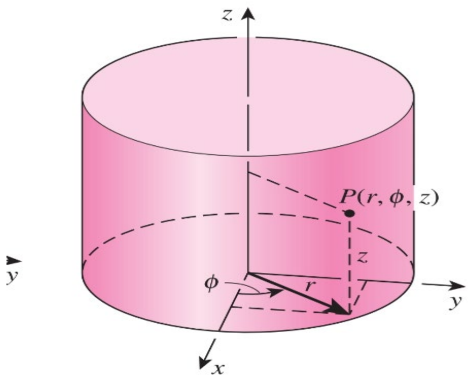


Figure 1. Cylindrical-coordinate (Yunus, 2003)

In the last two century, the focus on research of the time-fractional partial differential equations (FPDEs) in areas of sciences and engineering have gained a great significance in the modelling of real-life problems, analysis of mathematical and physical sciences, applied physics which providing solutions to complex processes in applied sciences and engineering (Falade et al.,2023; Basim, 2019; Hassan, 2022; Jassim, 2016). Several researchers have employed the concepts of Fractional calculus in areas like Visco-elasticity, biology, electronics, signal processing, genetic algorithms, robotic technology, traffic systems, telecommunication, chemistry, physics, economics and finance, and a host of many areas in applied sciences (Sedeeg, 2016; Zhang et al., 2014; Mainardi, 2010).
The study of the time-fractional heat equation has attracted a lot of attention from many Mathematicians all over the world each trying to study and assess the governing parameters, coordinates, and or equations associated with the general heat conduction equation. For example, Khalid et al., (2020) studied time-fractional heat equation in general orthogonal curvilinear and cylindrical coordinate systems, Xiaoyun & Mingyu (2010) obtained solutions to a time-fractional diffusion-wave equation in cylindrical coordinates, Ibrahim et al., (2013) established a new difference scheme for time-fractional heat equations using Crank-Nicholson method, Dimple et al., (2020) proposed and applied the concept of the fractional heat equation in higher space dimensions, Luis et al., (2011) obtained and analyzed a set of generalized fractional heat equations in the second law of thermodynamics, Arshad et al., (2021) obtained the numerical solution of the heat equation in polar cylindrical coordinates, Hami & Ömer (2021) presented approximate numerical solutions of the fractional heat equation with heat source and loss, and Qutaiba et al., (2023) obtained gives a determination of time-dependent coefficient in time-fractional heat equation.
This paper is aim to extend and apply the work Falade et al., (2023) for the computational assessment of heat generation constant $\dot{e}_{gen[j]}$ on three-dimensional cylindrical time-fractional nonlinear heat conduction equation in a cylindrical coordinate and vary the fractional order of the equation (1) and study the relationship between the constant $\dot{e}_{gen[j]}$ and time-fractional order. The proposed algorithm aims to reduce the time taken and computational cost drastically while the numeric analytic solutions are achieved and the changes in $\dot{e}_{gen}$ characteristics are evaluated with the 3D cylindrical graphical representations.

PRELIMINARIES

Definition 1.
The Riemann-Liouville, fractional integral in three-dimensional space is defined as Yubin (2019).

$$ {}_0^RD_t^{-\alpha}\psi(r,\varphi,z,t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\psi(s,r,\varphi,z)\,ds, \quad \alpha>0, \qquad (2) $$

Suppose $\alpha = 1$ then equation (2) reduces to the following definite integral of the form:

$$ {}_0^RD_t^{-1}\psi(t,r,\varphi,z) = \int_0^t \psi(t,r,\varphi,z)\,ds, \quad \alpha = 1, \qquad (3) $$

DESCRIPTION OF THE ALGORITHM

To understand the basic concept of this algorithm, we considered Riemann-Liouville fractional definition 1 in Maple software coded and evaluated the couple time-fraction nonlinear equation (2) as follows:

Step 1:

Restart:
With (PDE tools):
With(plot):
$Digits := R^+;$
$N := R^+;$
$\psi(r,\varphi,z,0) = f(r,\varphi,z);$
$\psi_t(r,\varphi,z,0) = g(r,\varphi,z);$
$\psi[0] := f(r,\varphi,z) + t * g(r,\varphi,z);$

Step 2:

for n from 0 to N do

$$\boldsymbol{FPDE} := \beta * \left(\psi[n] * diff(\psi[n],[r,r]) + diff(\psi[n],[\varphi,\varphi]) + diff(\psi[n],[z,z]) + \frac{\dot{e}_{gen[j]}}{k}\right);$$

$$\psi[n+1] := simplify\left(\frac{1}{\mathrm{GAMMA}(\alpha)} * int\left((t-s)^{\alpha-1} * sub\left(\begin{matrix} s=t, \\ \boldsymbol{FPDE}\end{matrix}\right), s = 0 \ldots t\right), assume = nonnegative;$$

end do;

Step 3:

$$\boldsymbol{Sol1} := sum(\psi[k], k = 0 \ldots N+1)$$

Step 4:

for i from 0 by 0.2 to 2 do

$$\psi[i] := evalf\left(eval\left(\begin{matrix} Sol1, [t=0.1, r=i, \\ \varphi = i, z = i, \dot{e}_{gen[j]} = \begin{pmatrix}10\\20\\30\\40\\50\end{pmatrix} \\ \beta = 0.3, \alpha = 1.0]\end{matrix}\right)\right);$$

end do;

$$\psi[i] := evalf\left(eval\left(\begin{matrix} Sol1, [t=0.1, r=i, \\ \varphi = i, z = i, \dot{e}_{gen[j]} = \begin{pmatrix}10\\20\\30\\40\\50\end{pmatrix} \\ \beta = 0.3, \alpha = 0.8]\end{matrix}\right)\right);$$

end do;

⋮

$$\psi[i] := evalf\left(eval\left(\begin{matrix} Sol1, [t=0.1, r=i, \\ \varphi = i, z = i, \dot{e}_{gen[j]} = \begin{pmatrix}10\\20\\30\\40\\50\end{pmatrix} \\ \beta = 0.3, \alpha = 0.2]\end{matrix}\right)\right);$$

end do;

Step 5:

$$\psi[1.0] := evl(Sol1, [t = 0.1, z = i, \dot{e}_{gen[j]} = 10, \beta = 0.3, \alpha = 1.0]);$$

$$\psi[0.8] := evl(Sol1, [t = 0.1, z = i, \dot{e}_{gen[j]} = 10, \beta = 0.3, \alpha = 0.8]);$$

$$\psi[0.6] := evl(Sol1, [t = 0.1, z = i, \dot{e}_{gen[j]} = 10, \beta = 0.3, \alpha = 0.6]);$$

$$\psi[0.4] := evl(Sol1, [t = 0.1, z = i, \dot{e}_{gen[j]} = 10, \beta = 0.3, \alpha = 0.4]);$$

$$\psi[0.2] := evl(Sol1, [t = 0.1, z = i, \dot{e}_{gen[j]} = 10, \beta = 0.3, \alpha = 0.2]);$$

$$plot3d\left(\psi[1.0], r = -\frac{1}{\pi} \ldots \frac{1}{\pi}, \varphi = -\frac{1}{\pi} \ldots \frac{1}{\pi}, grid = [100, 100, color = blue]\right);$$

$$plot3d\left(\psi[0.8], r = -\frac{1}{\pi} \ldots \frac{1}{\pi}, \varphi = -\frac{1}{\pi} \ldots \frac{1}{\pi}, grid = [100, 100, color = red]\right);$$

$$plot3d\left(\psi[0.6], r = -\frac{1}{\pi} \ldots \frac{1}{\pi}, \varphi = -\frac{1}{\pi} \ldots \frac{1}{\pi}, grid = [100, 100, color = green]\right);$$

$$plot3d\left(\psi[0.4], r = -\frac{1}{\pi} \ldots \frac{1}{\pi}, \varphi = -\frac{1}{\pi} \ldots \frac{1}{\pi}, grid = [100, 100, color = purple]\right);$$

$$plot3d\left(\psi[0.2], r = -\frac{1}{\pi} \ldots \frac{1}{\pi}, \varphi = -\frac{1}{\pi} \ldots \frac{1}{\pi}, grid = [100, 100, color = yellow]\right);$$

Output: See Table 2 and Figures (2 to 6)

NUMERICAL EXPERIMENT

In this section, we present five test cases of the heat generation constant $\dot{e}_{gen}$ for the determination of the amount of heat generated in the medium using the proposed five-step algorithm.

Table 1. Experimental parameters of the rate of heat generation $\dot{e}_{gen[j]}$, β thermal diffusivity, and t coordinate.

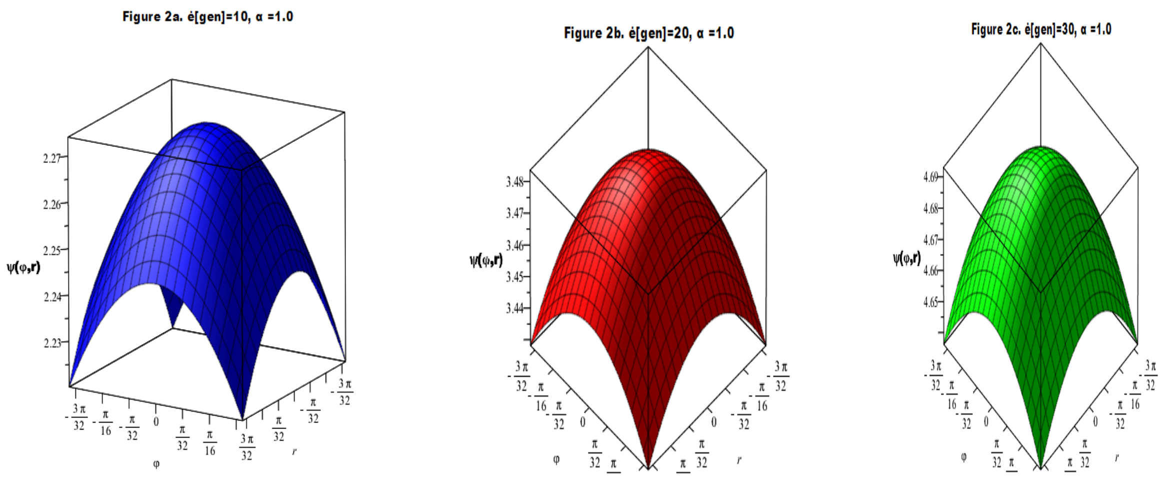
Parameters	Case 1	Case 2	Case 3	Case 4	Case 5
$\dot{e}_{gen}$	10	20	30	40	50
β	0.3	0.3	0.3	0.3	0.3
t	0.1	0.1	0.1	0.1	0.1
k	1.0	1.0	1.0	1.0	1.0

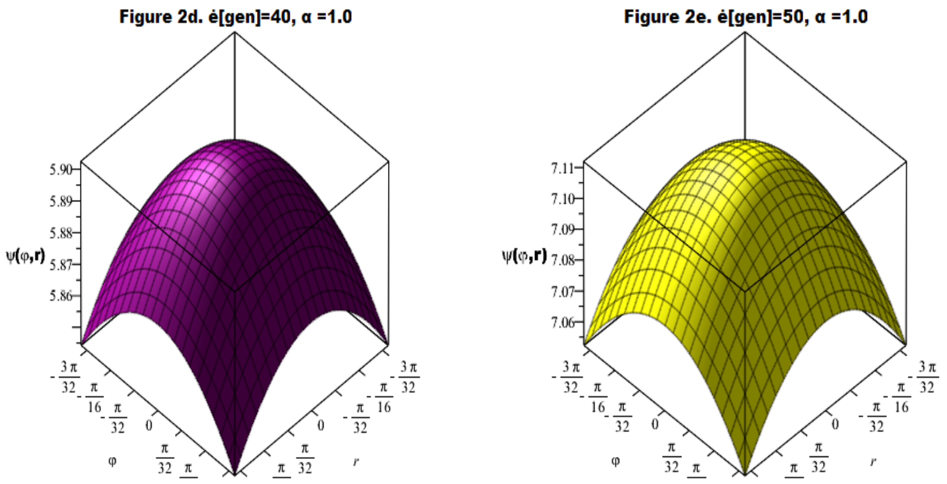
Table 2. Simulation results for the heat constant $\dot{e}_{gen[j]}$ on the variation of the time-fractional nonlinear heat equation in a cylindrical coordinate.

(r, φ, z)	Order	Case 1	Case 2	Case 3	Case 4	Case 5
(0,0,0)	$\alpha = 1.0$	1.624717507	2.225856907	2.826996307	3.428135707	4.029275107
	$\alpha = 0.8$	2.004607157	3.031213962	4.057820768	5.084427575	6.111034381
	$\alpha = 0.6$	2.629726937	4.341785713	6.053844487	7.765903267	9.477962046
	$\alpha = 0.4$	3.643724683	6.438794175	9.233863665	12.02893316	14.82400264
	$\alpha = 0.2$	5.273264555	9.766161316	14.25905808	18.75195484	23.24485159
(0.2,0.2,0.2)	$\alpha = 1.0$	1.578803755	2.179834625	2.780865495	3.381896365	3.982927236
	$\alpha = 0.8$	1.962261838	2.988333878	4.014405918	5.040477958	6.066549998
	$\alpha = 0.6$	2.590982962	4.300630481	6.010278000	7.719925521	9.429573041
	$\alpha = 0.4$	3.604814768	6.390080532	9.175346296	11.96061206	14.74587782
	$\alpha = 0.2$	5.219832796	9.677510148	14.13518750	18.59286485	23.05054220
(0.4,0.4,0.4)	$\alpha = 1.0$	1.456354435	2.057125243	2.657896051	3.258666858	3.859437666
	$\alpha = 0.8$	1.849446032	2.874236654	3.899027276	4.923817897	5.948608518
	$\alpha = 0.6$	2.488316391	4.192185999	5.896055605	7.599925212	9.303794818
	$\alpha = 0.4$	3.503315121	6.265088957	9.026862789	11.78863662	14.55041046
	$\alpha = 0.2$	5.083376354	9.456660111	13.82994387	18.20322763	22.57651139
(0.6,0.6,0.6)	$\alpha = 1.0$	1.300219148	1.900722802	2.501226456	3.101730110	3.702233765
	$\alpha = 0.8$	1.706617566	2.730091826	3.753566085	4.777040345	5.800514603
	$\alpha = 0.6$	2.361016075	4.058950210	5.756884347	7.454818483	9.152752619
	$\alpha = 0.4$	3.383814226	6.121455523	8.859096824	11.59673812	14.33437942
	$\alpha = 0.2$	4.933751753	9.220340567	13.50692939	17.79351820	22.08010702
(0.8,0.8,0.8)	$\alpha = 1.0$	1.172456259	1.772808879	2.373161500	2.973514120	3.573866740
	$\alpha = 0.8$	1.593886219	2.616616280	3.639346343	4.662076405	5.684806466
	$\alpha = 0.6$	2.268861205	3.963439755	5.658018305	7.352596856	9.047175407
	$\alpha = 0.4$	3.313596153	6.037594258	8.761592362	11.48559047	14.20958857
	$\alpha = 0.2$	4.870505662	9.108081974	13.34565830	17.58323461	21.82081092
(1.0,1.0,1.0)	$\alpha = 1.0$	1.143966950	1.744346564	2.344726178	2.945105792	3.545485406
	$\alpha = 0.8$	1.582472698	2.605335768	3.628198838	4.651061909	5.673924977
	$\alpha = 0.6$	2.285434179	3.980612462	5.675790744	7.370969028	9.066147309
	$\alpha = 0.4$	3.375629389	6.102065895	8.828502402	11.55493891	14.28137541
	$\alpha = 0.2$	5.006683046	9.253019195	13.49935535	17.74569150	21.99202765
(1.2,1.2,1.2)	$\alpha = 1.0$	1.289633468	1.890295382	2.490957296	3.091619210	3.692281124
	$\alpha = 0.8$	1.752323232	2.776577293	3.800831356	4.825085418	5.849339479

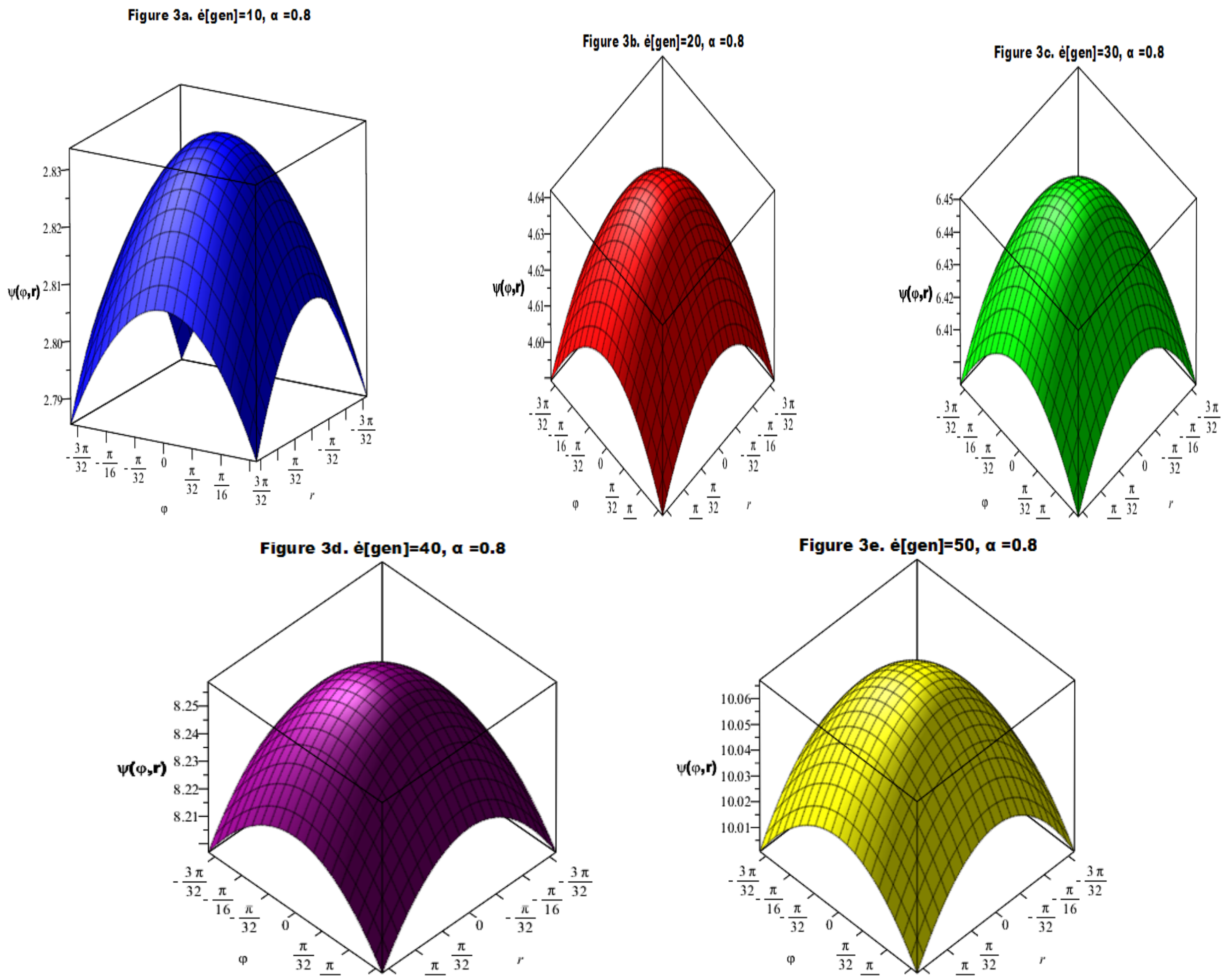
	$\alpha = 0.6$	2.501273941	4.202724203	5.904174463	7.605624724	9.307074985
	$\alpha = 0.4$	3.684727480	6.436664697	9.188601913	11.94053913	14.69247635
	$\alpha = 0.2$	5.515826332	9.853772525	14.19171873	18.52966492	22.86761112
(1.4,1.4,1.4)	$\alpha = 1.0$	1.701002622	2.302496024	2.903989426	3.505482828	4.106976230
	$\alpha = 0.8$	2.211003707	3.239354809	4.267705909	5.296057010	6.324408110
	$\alpha = 0.6$	3.058982064	4.778905861	6.498829657	8.218753455	9.938677250
	$\alpha = 0.4$	4.467405117	7.294452333	10.12149955	12.94854677	15.77559398
	$\alpha = 0.2$	6.834276958	11.44205212	16.04982728	20.65760244	25.26537759
(1.6,1.6,1.6)	$\alpha = 1.0$	2.534952750	3.138805555	3.742658360	4.346511165	4.950363970
	$\alpha = 0.8$	3.163188615	4.203165335	5.243142054	6.283118774	7.323095495
	$\alpha = 0.6$	4.278113479	6.050457158	7.822800837	9.595144517	11.36748820
	$\alpha = 0.4$	6.346422983	9.386599805	12.42677663	15.46695345	18.50713027
	$\alpha = 0.2$	10.40385572	15.77728845	21.15072119	26.52415393	31.89758665
(1.8,1.8,1.8)	$\alpha = 1.0$	4.139176318	4.750010061	5.360843804	5.971677547	6.582511290
	$\alpha = 0.8$	5.119926074	6.194300365	7.268674659	8.343048952	9.417423244
	$\alpha = 0.6$	7.116447866	9.043890097	10.97133232	12.89877456	14.82621679
	$\alpha = 0.4$	11.60758312	15.27836214	18.94914115	22.61992017	26.29069919
	$\alpha = 0.2$	22.42051565	30.05935562	37.69819558	45.33703555	52.97587553
(2.0,2.0,2.0)	$\alpha = 1.0$	7.402216739	8.034585651	8.666954562	9.299323474	9.931692385
	$\alpha = 0.8$	9.681020933	10.86150668	12.04199243	13.22247817	14.40296392
	$\alpha = 0.6$	15.48086368	17.88676214	20.29266060	22.69855906	25.10445754
	$\alpha = 0.4$	31.92861127	37.54470538	43.16079950	48.77689359	54.39298770
	$\alpha = 0.2$	79.25952077	93.88680907	108.5140973	123.1413856	137.7686739

Figures 2 (a, b, c,d,e): The graphical representation of heat generation constant$\dot{e}_{gen} = [10, 20, 30, 40, 50]$ and $\alpha = 1.0$

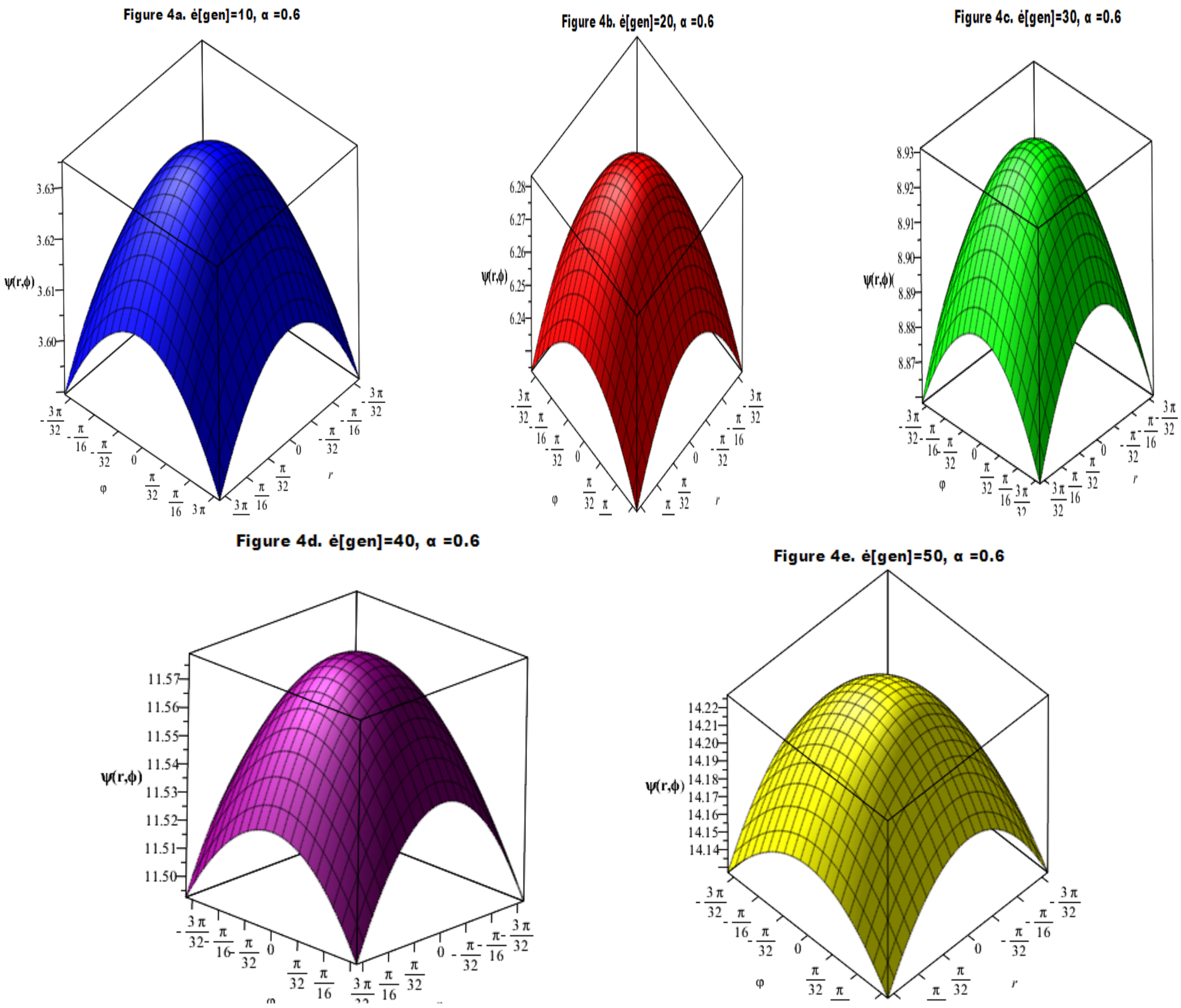




Figures3 (a, b,c,d,e). The graphical representation of heat generation constant$\dot{e}_{gen} = [10, 20, 30, 40, 50]$ and $\alpha = 0.8$



Figures4 (a,b,c,d,e): The graphical representation of heat generation constant $\dot{e}_{gen} = [10, 20, 30, 40, 50]$and $\alpha = 0.6$



Figures5 (a,b,c,d,e): The graphical representation of heat generation constant $\dot{e}_{gen} = [10, 20, 30, 40, 50]$and $\alpha = 0.4$

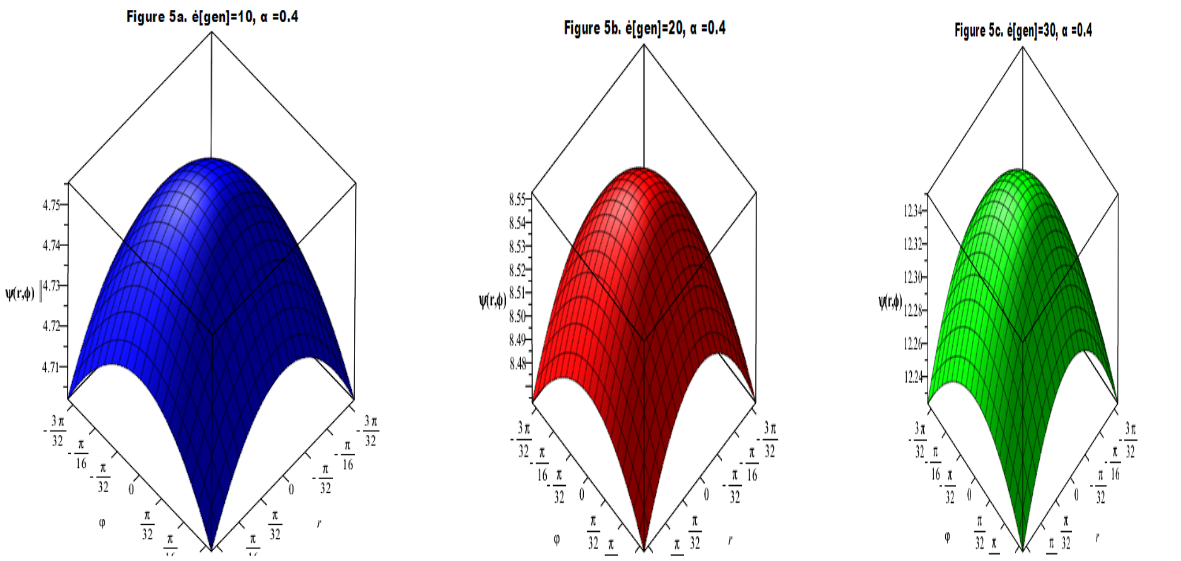


Figure 5d. ė[gen]=40, α =0.4

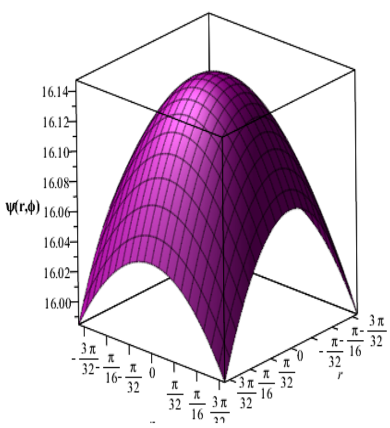
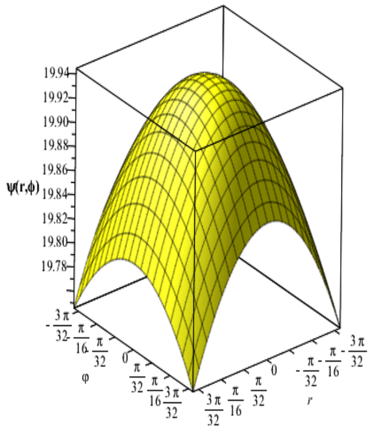


Figure 5e. ė[gen]=50, α =0.4



Figures6 (a,b,c,d,e): The graphical representation of heat generation constant $\dot{e}_{gen} = [10, 20, 30, 40, 50]$and $\alpha = 0.2$

Figure 6a. ė[gen]=10, α =0.2

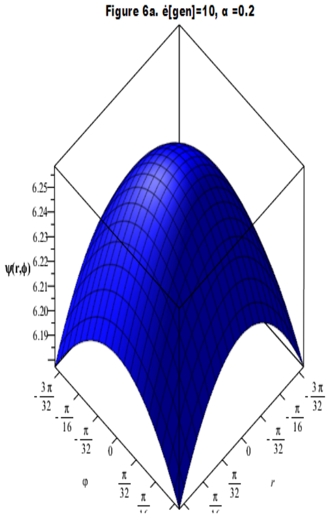


Figure 6b. ė[gen]=20, α =0.2

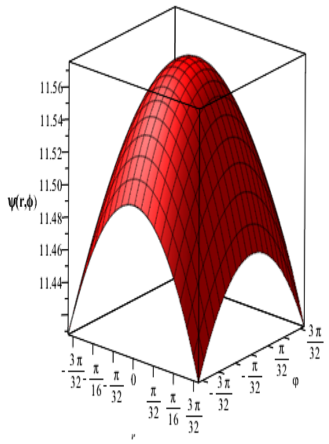


Figure 6c. ė[gen]=30, α =0.2

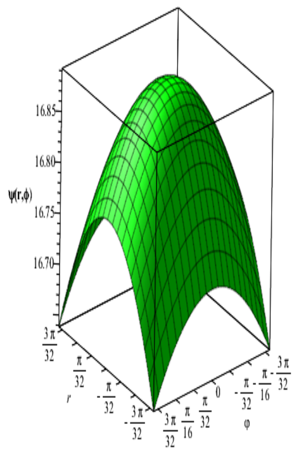


Figure 6d. ė[gen]=40, α =0.2

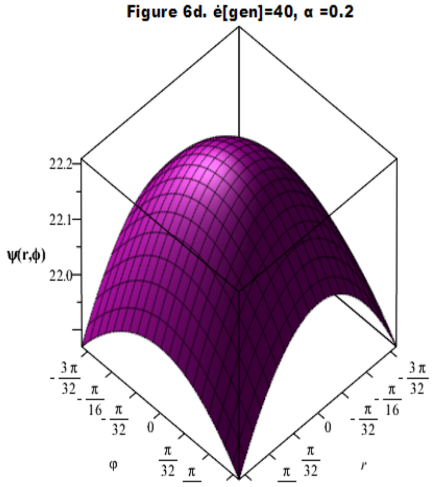
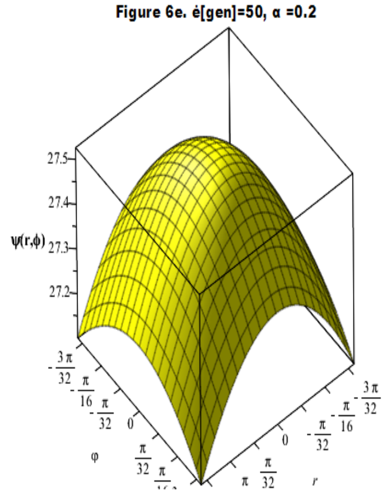


Figure 6e. ė[gen]=50, α =0.2



RESULTDISCUSSION

Figures 2. Depicts the simulation results obtained when the heat generation constants $\dot{e}_{gen} = \mathbf{10, 20, 30, 40, 50}$ and the fraction order is integer $= \mathbf{1.0}$. From the computational results obtained the following maximum heat is generated when at the$\boldsymbol{\alpha} = \mathbf{1.0}$

Figure 2a $\dot{e}_{gen} = \mathbf{10}\ (\mathbf{2.27})$
Figure 2b $\dot{e}_{gen} = \mathbf{20}\ (\mathbf{3.48})$
Figure 2c $\dot{e}_{gen} = \mathbf{30}\ (\mathbf{4.49})$
Figure 2d $\dot{e}_{gen} = \mathbf{40}\ (\mathbf{5.90})$
Figure 2e $\dot{e}_{gen} = \mathbf{50}\ (\mathbf{7.11})$

Figures 3. Depict the simulation results obtained when the heat generation constants $\dot{e}_{gen} = \mathbf{10, 20, 30, 40, 50}$, and the fraction order was $\boldsymbol{\alpha} = \mathbf{0.8}$. From the computational results obtained the following maximum heat is generated:

Figure 3a $\dot{e}_{gen} = \mathbf{10}\ (\mathbf{2.83})$
Figure 3b $\dot{e}_{gen} = \mathbf{20}\ (\mathbf{4.64})$
Figure 3c $\dot{e}_{gen} = \mathbf{30}\ (\mathbf{6.45})$
Figure 3d $\dot{e}_{gen} = \mathbf{40}\ (\mathbf{8.25})$
Figure 3e $\dot{e}_{gen} = \mathbf{50}\ (\mathbf{10.06})$

Figures 4. Depict the simulation results obtained when the heat generation constants $\dot{e}_{gen} = \mathbf{10, 20, 30, 40, 50}$, and the fraction order was $\boldsymbol{\alpha} = \mathbf{0.6}$ from the computational results obtained the following maximum heat is generated:

Figure 4a $\dot{e}_{gen} = \mathbf{10}\ (\mathbf{3.63})$
Figure 4b $\dot{e}_{gen} = \mathbf{20}\ (\mathbf{6.28})$
Figure 4c $\dot{e}_{gen} = \mathbf{30}\ (\mathbf{8.93})$
Figure 4d $\dot{e}_{gen} = \mathbf{40}\ (\mathbf{11.57})$
Figure 4e $\dot{e}_{gen} = \mathbf{50}\ (\mathbf{14.22})$

Figures 5. Depict the simulation results obtained when the heat generation constants $\dot{e}_{gen} = \mathbf{10, 20, 30, 40, 50}$, and the fraction order was $\boldsymbol{\alpha} = \mathbf{0.4}$ From the computational results obtained the following maximum heat is generated:

Figure 5a $\dot{e}_{gen} = \mathbf{10}\ (\mathbf{4.75})$
Figure 5b $\dot{e}_{gen} = \mathbf{20}\ (\mathbf{8.55})$
Figure 5c $\dot{e}_{gen} = \mathbf{30}\ (\mathbf{12.34})$
Figure 5d $\dot{e}_{gen} = \mathbf{40}\ (\mathbf{16.14})$
Figure 5e $\dot{e}_{gen} = \mathbf{50}\ (\mathbf{19.94})$

Figures 6. Depict the simulation results obtained when the heat generation constants $\dot{e}_{gen} = \mathbf{10, 20, 30, 40, 50}$, and the fraction order was $\boldsymbol{\alpha} = \mathbf{0.2}$ From the computational results obtained the following maximum heat is generated:

Figure 6a $\dot{e}_{gen} = \mathbf{10}\ (\mathbf{6.25})$
Figure 6b $\dot{e}_{gen} = \mathbf{20}\ (\mathbf{11.56})$
Figure 6c $\dot{e}_{gen} = \mathbf{30}\ (\mathbf{16.85})$
Figure 6d $\dot{e}_{gen} = \mathbf{40}\ (\mathbf{22.20})$
Figure 6e $\dot{e}_{gen} = \mathbf{50}\ (\mathbf{27.55})$

Finally, the assessment of the heat generation constant $\dot{e}_{gen}$demonstrated that higher the heat constant, yielded more heat obtainedirrespective of the order of the three-dimensional time-fractional non-linear heat equation in a cylindrical coordinate.

CONCLUSION

In this article, the heat generation constant$\dot{e}_{gen}$ on time fractional nonlinear three-dimensional heat equation in cylindrical coordinates was investigated utilizing a newly formulated five-step algorithm. Applying this technique, we have successfully obtained a closed analytical solution which embedded with time-fractional order and the rate of heart generation constant using Caputo definition in the Maple software coded form. The simulation solutions are carried out by increasing the rate of the heat generation constant$\dot{e}_{gen}$. The results obtained are presented in 3D plots which depict the increase in heat generation constant and yielded an increase of heat in the medium irrespective of the order of the heat equation. The simulation results represent a significant contribution to the understanding and exploration of the software package in solving time-fractional equations. The implemented method was straightforward, trustworthy, and efficient, leading to significant change in the field of computational mathematics. Therefore, the present approach paves the way for further applications to investigate linear and nonlinear time-fraction evolution equations that commonly appear in various areas of applied sciences and engineering.

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APPENDIX

Case 1: $\dot{e}_{gen} = 10, \beta = 0.3, t = 0.1$

Step 1:

```
restart :
with(PDEtools) :
with(linalg) :
with(plots) :
Digits := 12 :
N := 2;
```

```
ψ[0, r, φ, z] := (cos(r)·cos(φ)·cos(z)) + t·(cosh(r)·cosh(φ)
    ·cosh(z))
```

$$\psi_{0, r, \varphi, z} := \cos(r)\cos(\varphi)\cos(z) + t\cosh(r)\cosh(\varphi)\cosh(z)$$

```
ψ[0] := ψ[0, r, φ, z];
```

$$\psi_0 := \cos(r)\cos(\varphi)\cos(z) + t\cosh(r)\cosh(\varphi)\cosh(z)$$

Step 2:

```
or n from 0 to N do
 F := β·(ψ[n]·diff(ψ[n], r, r) + diff(ψ[n], φ, φ) + diff(ψ[n], z, z)
     + ė_gen);
 ψ[n + 1] := simplify(1/GAMMA(α)·int((t − s)^(α − 1)·F, s = 0..t),
     'assume = nonnegative') :
end do:
```

Step 3:

```
Sol1 := sum(ψ[k], k = 0..N + 1);
```

$$
\begin{aligned}
&\cos(r)\cos(\varphi)\cos(z) + t\cosh(r)\cosh(\varphi)\cosh(z) - \frac{1}{\Gamma(\alpha+1)}\Big(t^{\alpha}\beta\big(\\
&-\cosh(r)^2\cosh(\varphi)^2\cosh(z)^2 t^2 + \cos(r)^2\cos(\varphi)^2\cos(z)^2\\
&-2t\cosh(r)\cosh(\varphi)\cosh(z) + 2\cos(r)\cos(\varphi)\cos(z) - \dot{e}\_gen\big)\Big)\\
&+\frac{1}{\Gamma(\alpha+1)^3}\Big(\beta\big(2t^{3\alpha+2}\beta^2\cos(\varphi)^2\cos(r)^2\cos(z)^2\cosh(\varphi)^2\\
&\cosh(z)^2\\
&-6t^{3\alpha+2}\beta^2\cos(\varphi)\cos(r)\cos(z)\cosh(\varphi)^2\cosh(r)^2\cosh(z)^2\\
&+6t^{3\alpha+1}\beta^2\cos(\varphi)^2\cos(r)^2\cos(z)^2\cosh(\varphi)\cosh(r)\cosh(z)\\
&+t^{\alpha}\dot{e}\_gen\,\Gamma(\alpha+1)^2 + 4t^{3\alpha+4}\beta^2\cosh(\varphi)^4\cosh(r)^4\cosh(z)^4\\
&+10t^{3\alpha+3}\beta^2\cosh(\varphi)^3\cosh(r)^3\cosh(z)^3\\
&+4t^{3\alpha+2}\beta^2\cosh(\varphi)^2\cosh(r)^2\cosh(z)^2\\
&-4t^{3\alpha}\beta^2\cos(\varphi)^2\cos(r)^2\cos(z)^2 - 4t^{3\alpha}\beta^2\cos(\varphi)^4\cos(r)^4\cos(z)^4\\
&-10t^{3\alpha}\beta^2\cos(\varphi)^3\cos(r)^3\cos(z)^3\\
&-2t^{2\alpha+2}\beta\cosh(\varphi)^2\cosh(r)^2\Gamma(\alpha+1)\\
&-2t^{2\alpha}\beta\cos(\varphi)^2\cos(r)^2\Gamma(\alpha+1) - 2t^{2\alpha}\beta\cos(r)^2\cos(z)^2\Gamma(\alpha+1)\\
&-2t^{2\alpha+2}\beta\cosh(r)^2\cosh(z)^2\Gamma(\alpha+1)\\
&-2t^{3\alpha+2}\beta^2\dot{e}\_gen\cosh(\varphi)^2\cosh(z)^2\\
&+2t^{3\alpha}\beta^2\cos(z)^4\cos(\varphi)^4\cos(r)^2 + 4t^{3\alpha}\beta^2\cos(z)^3\cos(\varphi)^3\cos(r)\\
&-2t^{3\alpha}\beta^2\dot{e}\_gen\cos(\varphi)^2\cos(z)^2\\
&-2t^{3\alpha+4}\beta^2\cosh(z)^4\cosh(\varphi)^4\cosh(r)^2\\
&-4t^{3\alpha+3}\beta^2\cosh(z)^3\cosh(\varphi)^3\cosh(r)\\
&+4t^{2\alpha+1}\beta\cosh(\varphi)\cosh(r)\cosh(z)\Gamma(\alpha+1)\\
&+8t^{2\alpha}\beta\cos(\varphi)^2\cos(r)^2\cos(z)^2\Gamma(\alpha+1)\\
&+4t^{2\alpha}\beta\cos(\varphi)\cos(r)\cos(z)\Gamma(\alpha+1)\\
&+8t^{2\alpha+2}\beta\cosh(\varphi)^2\cosh(r)^2\cosh(z)^2\Gamma(\alpha+1)\\
&+4t^{3\alpha+2}\beta^2\dot{e}\_gen\cosh(\varphi)^2\cosh(r)^2\cosh(z)^2\\
&+2t^{3\alpha+1}\beta^2\dot{e}\_gen\cosh(\varphi)\cosh(r)\cosh(z)\\
&+4t^{3\alpha}\beta^2\dot{e}\_gen\cos(\varphi)^2\cos(r)^2\cos(z)^2\\
&+2t^{3\alpha}\beta^2\dot{e}\_gen\cos(\varphi)\cos(r)\cos(z)\\
&-4t^{3\alpha+1}\beta^2\cos(\varphi)^2\cos(z)^2\cosh(\varphi)\cosh(r)\cosh(z)\\
&+4t^{3\alpha+2}\beta^2\cos(\varphi)\cos(r)\cos(z)\cosh(\varphi)^2\cosh(z)^2\\
&-2t^{3\alpha+2}\beta^2\cos(\varphi)^2\cos(z)^2\cosh(\varphi)^2\cosh(r)^2\cosh(z)^2\big)\Big)
\end{aligned}
$$

Step 4:

```
for i from 0 by 0.2 to 2 do
 ψ[i] := evalf(eval(Sol1, [t = 0.1, r = i, φ = i, z = i, ė_gen = 10, β = 0.3, α
    = 1])) :
 end do;
```

$\psi_{0} := 1.62471750740$

$\psi_{0.2} := 1.57880375502$

$\psi_{0.4} := 1.45635443462$

$\psi_{0.6} := 1.30021914763$

$\psi_{0.8} := 1.17245625947$

$\psi_{1.0} := 1.14396694990$

$\psi_{1.2} := 1.28963346815$

$\psi_{1.4} := 1.70100262205$

$\psi_{1.6} := 2.53495275030$

$\psi_{1.8} := 4.13917631913$

$\psi_{2.0} := 7.40221673831$

```
for i from 0 by 0.2 to 2 do
 ψ[i] := evalf(eval(Sol1, [t = 0.1, r = i, φ = i, z = i, ė_gen = 10, β = 0.3, α
    = 0.8])) :
 end do;
```

$\psi_{0} := 2.00460715676$

$\psi_{0.2} := 1.96226183797$

$\psi_{0.4} := 1.84944603264$

$\psi_{0.6} := 1.70661756577$

$\psi_{0.8} := 1.59388621921$

$\psi_{1.0} := 1.58247269774$

$\psi_{1.2} := 1.75232323285$

$\psi_{1.4} := 2.21100370877$

$\psi_{1.6} := 3.16318861568$

$\psi_{1.8} := 5.11992607641$

$\psi_{2.0} := 9.68102093127$

```
for i from 0 by 0.2 to 2 do
 ψ[i] := evalf(eval(Sol1, [t = 0.1, r = i, φ = i, z = i, ė_gen = 10, β = 0.3, α
    = 0.6])) :
 end do;
```

$\psi_{0} := 2.62972693610$

$\psi_{0.2} := 2.59098296184$

$\psi_{0.4} := 2.48831639084$

$\psi_{0.6}$:= 2.36101607515
$\psi_{0.8}$:= 2.26886120562
$\psi_{1.0}$:= 2.28543417870
$\psi_{1.2}$:= 2.50127394133
$\psi_{1.4}$:= 3.05898206480
$\psi_{1.6}$:= 4.27811348007
$\psi_{1.8}$:= 7.11644786765
$\psi_{2.0}$:= 15.4808636709

```
for i from 0 by 0.2 to 2 do
 ψ[i] := evalf(eval(Sol1, [t = 0.1, r = i, φ = i, z = i, ė_gen = 10, β = 0.3, α = 0.4])) :
end do;
```

ψ_{0} := 3.64372468269
$\psi_{0.2}$:= 3.60481476756
$\psi_{0.4}$:= 3.50331512161
$\psi_{0.6}$:= 3.38381422569
$\psi_{0.8}$:= 3.31359615339
$\psi_{1.0}$:= 3.37562938829
$\psi_{1.2}$:= 3.68472748085
$\psi_{1.4}$:= 4.46740511731
$\psi_{1.6}$:= 6.34642298366
$\psi_{1.8}$:= 11.6075831214
$\psi_{2.0}$:= 31.9286112577

```
for i from 0 by 0.2 to 2 do
 ψ[i] := evalf(eval(Sol1, [t = 0.1, r = i, φ = i, z = i, ė_gen = 10, β = 0.3, α = 0.2])) :
end do;
```

ψ_{0} := 5.27326455378
$\psi_{0.2}$:= 5.21983279460
$\psi_{0.4}$:= 5.08337635204
$\psi_{0.6}$:= 4.93375175208
$\psi_{0.8}$:= 4.87050566138
$\psi_{1.0}$:= 5.00668304469
$\psi_{1.2}$:= 5.51582633178
$\psi_{1.4}$:= 6.83427695895
$\psi_{1.6}$:= 10.4038557193
$\psi_{1.8}$:= 22.4205156653
$\psi_{2.0}$:= 79.2595207333

Step 5:

```
ψ[1.0] := eval(Sol1, [t = 0.1, z = i, ė_gen = 10, β = 0.3, α = 1.0]) :
ψ[0.8] := eval(Sol1, [t = 0.1, z = i, ė_gen = 10, β = 0.3, α = 0.8]) :
ψ[0.6] := eval(Sol1, [t = 0.1, z = i, ė_gen = 10, β = 0.3, α = 0.6]) :
ψ[0.4] := eval(Sol1, [t = 0.1, z = i, ė_gen = 10, β = 0.3, α = 0.4]) :
ψ[0.2] := eval(Sol1, [t = 0.1, z = i, ė_gen = 10, β = 0.3, α = 0.2]) :
```

$$plot3d\left(\psi[1.0], r = -\frac{1}{\pi}..\frac{1}{\pi}, \varphi = -\frac{1}{\pi}..\frac{1}{\pi}, grid = [100, 100], color = \text{"blue"}\right);$$

$$plot3d\left(\psi[0.8], r = -\frac{1}{\pi}..\frac{1}{\pi}, \varphi = -\frac{1}{\pi}..\frac{1}{\pi}, grid = [100, 100], color = \text{"red"}\right);$$

$$plot3d\left(\psi[0.6], r = -\frac{1}{\pi}..\frac{1}{\pi}, \varphi = -\frac{1}{\pi}..\frac{1}{\pi}, grid = [100, 100], color = \text{"green"}\right);$$

$$plot3d\left(\psi[0.4], r = -\frac{1}{\pi}..\frac{1}{\pi}, \varphi = -\frac{1}{\pi}..\frac{1}{\pi}, grid = [100, 100], color = \text{"purple"}\right);$$

$$plot3d\left(\psi[0.2], r = -\frac{1}{\pi}..\frac{1}{\pi}, \varphi = -\frac{1}{\pi}..\frac{1}{\pi}, grid = [100, 100], color = \text{"yellow"}\right);$$